EARTHQUAKE ENGINEERING AND STRUCTURAL DYNAMICS
Earthquake Engng. Struct. Dyn. **27**, 609–618 (1998)

RECURSIVE EVALUATION OF INTERACTION FORCES AND PROPERTY MATRICES FROM UNIT-IMPULSE RESPONSE FUNCTIONS OF UNBOUNDED MEDIUM BASED ON BALANCING APPROXIMATION

ANTONIO PARONESSO AND JOHN P. WOLF*
Department of Civil Engineering, Institute of Hydraulics and Energy, Swiss Federal Institute of Technology Lausanne, CH-1015 Lausanne, Switzerland

SUMMARY

Starting from the unit-impulse response matrix of the unbounded medium, a discrete-time formulation permitting the recursive evaluation of the interaction forces and a continuous-time formulation yielding property matrices corresponding to a model with a finite number of degrees of freedom are discussed. This is achieved using the balancing approximation method which is easily automated, guarantees stability and leads to highly accurate results. © 1998 John Wiley & Sons Ltd.

KEY WORDS: balancing approximation; recursive evaluation of interaction forces; property matrices; rational approximation; soil–structure interaction; unit-impulse response

1. INTRODUCTION

In a dynamic unbounded medium–structure-interaction problem based on the substructure method (Figure 1a), the bounded structure (which can include an adjacent irregular domain of the medium) and the unbounded (semi-infinite or infinite) medium are coupled on the structure–medium interface. In general, the structure will exhibit non-linear behaviour while the unbounded medium is always assumed to remain linear.

The interaction force $\{R(t)\}$-displacement $\{u(t)\}$ relationship with respect to the N degrees of freedom of the nodes on the structure–medium interface of the unbounded medium is global in space and time, Figure 1b:[1]

$$\{R(t)\} = [K_\infty]\{u(t)\} + [C_\infty]\{\dot{u}(t)\} + \int_0^t [S_r(t-\tau)]\{u(\tau)\}\,\mathrm{d}\tau \tag{1}$$

The first two terms on the right-hand side representing the instantaneous response denote the singular part with $[K_\infty]$ and $[C_\infty]$ defining the high-frequency limit ($\omega \to \infty$) of the dynamic-stiffness matrix $[S(\omega)]$. The third term describing the lingering response is equal to the regular part (subscript r) consisting of the convolution of the corresponding unit–impulse response function $[S_r(t)]$ and the displacement.

The scaled boundary finite-element method, alias the consistent infinitesimal finite-element cell method, determines $[K_\infty]$, $[C_\infty]$ and $[S_r(t_j)]$ at distinct time stations t_j efficiently.[2,3] Alternatively, the boundary-element procedure can be applied. Note that for computational efficiency, the number of time steps must be limited. Thus, partial data only are available ($t_j \leqslant t_{\max}$).

* Correspondence to: John P. Wolf, Department of Civil Engineering, Institute of Hydraulics and Energy, Swiss Federal Institute of Technology Lausanne, CH-1015 Lausanne, Switzerland

CCC 0098–8847/98/060609–10$17·50
Received 11 March 1997
Revised 16 September 1997

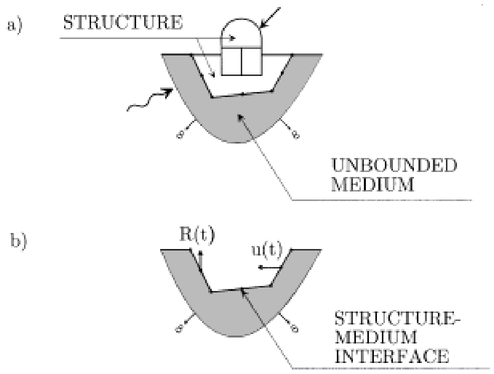

Figure 1. Problem definition of dynamic unbounded medium–structure-interaction analysis: (a) coupled system; (b) unbounded medium

The interaction forces at a specific time depend on the time histories of the displacements in all nodes from the start of the excitation onwards. In this rigorous formulation a large computational effort (proportional to the *square* of the number of time steps) and storage requirement result, which makes it unrealistic to evaluate large practical problems.

To reduce the computational effort, concepts of linear system theory can be applied. A systematic detailed description of the various procedures starting from either $[S_r(t_j)]$ or $[S_r(\omega_j)]$ and leading to either a continuous-time or discrete-time formulation is contained in Reference 4. All methods are based on a *rational approximation* of the frequency response of the unbounded medium. This corresponds to introducing some internal degrees of freedom in addition to those present on the structure–medium interface. The unbounded medium is thus represented as a model with a finite number of degrees of freedom, in the same way as the structure is. The computational effort becomes proportional to the number of time steps and the storage requirement is also greatly reduced.

In the discrete-time formulation, explicit finite difference or recursive equations are established and in the continuous-time formulation linear differential equations result. Both are referred to as a *realization* of the rational approximation.

Only the efficient method starting from $[S_r(t_j)]$ based on the so-called *balancing approximation* is addressed in this paper. As a result, the discrete-time formulation consisting of *finite-difference equations* of first order at the time step k follows as

$$\{x(k+1)\} = [\tilde{A}]\{x(k)\} + [\tilde{B}]\{u(k)\} \tag{2a}$$

$$\{R(k)\} = [\tilde{C}]\{x(k)\} + [\tilde{D}]\{u(k)\} + [K_\infty]\{u(k)\} + [C_\infty]\{\dot{u}(k)\} \tag{2b}$$

$\{x(k)\}$ represents the M state variables corresponding to the internal degrees of freedom. The matrices $[\tilde{A}]$, $[\tilde{B}]$, $[\tilde{C}]$ and $[\tilde{D}]$ are constant. Alternatively, as an equivalent discrete-time formulation, *recursive equations* expressing the regular part of the interaction forces at time step $k\{R_r(k)\}$ as a function of the M previous interaction forces $\{R_r(k-m)\}$, the displacement at step k and $\{u(k)\}$ and the M previous displacements $\{u(k-m)\}$ are written as

$$\{R_r(k)\} = -\sum_{m=1}^{M} [\tilde{a}_m]\{R_r(k-m)\} + \sum_{m=0}^{M} [\tilde{b}_m]\{u(k-m)\} \tag{3}$$

where

$$\{R(k)\} = \{R_r(k)\} + [K_\infty]\{u(k)\} + [C_\infty]\{\dot{u}(k)\} \tag{4}$$

applies. The matrices $[\tilde{a}_m]$ and $[\tilde{b}_m]$ are constant.

Starting from the discrete-time formulation of equation (2), a continuous-time description can be derived. This system of *linear differential equations of first order* equals

$$\{\dot{x}(t)\} = [A]\{x(t)\} + [B]\{u(t)\} \tag{5a}$$

$$\{R(t)\} = [C]\{x(t)\} + [K_\infty]\{u(t)\} + [C_\infty]\{\dot{u}(t)\} \tag{5b}$$

with the constant matrices $[A]$, $[B]$, and $[C]$. The terms associated with $[A]$, $[B]$, and $[C]$ represent the regular part and the terms associated with $[K_\infty]$ and $[C_\infty]$ the singular part. Note that in contrast to the discrete-time formulation in equation (2b) no term $[D]$ $\{u(t)\}$ occurs in an appropriate approximation of the regular part (which must be strictly proper) in the continuous-time formulation.

The non-symmetric system of equation (5) can be transformed to a system of *first-order differential equations with the symmetric damping and static-stiffness matrices* $[C^1]$ and $[K^1]$

$$[C^1]\begin{Bmatrix}\{\dot{x}(t)\}\\\{\dot{u}(t)\}\end{Bmatrix} + [K^1]\begin{Bmatrix}\{x(t)\}\\\{u(t)\}\end{Bmatrix} = \begin{Bmatrix}\{0\}\\\{R(t)\}\end{Bmatrix} \tag{6}$$

or to a system of *second-order differential equations with the symmetric mass, damping and static-stiffness matrices* $[M^2]$, $[C^2]$ and $[K^2]$

$$[M^2]\begin{Bmatrix}\{\ddot{w}(t)\}\\\{\ddot{u}(t)\}\end{Bmatrix} + [C^2]\begin{Bmatrix}\{\dot{w}(t)\}\\\{\dot{u}(t)\}\end{Bmatrix} + [K^2]\begin{Bmatrix}\{w(t)\}\\\{u(t)\}\end{Bmatrix} = \begin{Bmatrix}\{0\}\\\{R(t)\}\end{Bmatrix} \tag{7}$$

Typically, the order of the internal variables vector $\{w\}$ is half of that of $\{x\}$. Equation (7) for the model of the unbounded medium is in the same form as the equation of motion in finite-element analysis of a structure with a finite number of degrees of freedom. The static-stiffness, damping and mass matrices are called the property matrices.

It will become apparent that the described procedure of the balancing approximation can easily be automated. Selecting a tolerance yields the degree M of the rational approximation. In addition, the procedure always leads to stable realizations.

To establish the significance of the rational approximation in the time domain, equation (5a) is solved yielding

$$\{x(t)\} = \int_0^t e^{[A](t-\tau)}[B]\{u(\tau)\}\,d\tau \tag{8}$$

where $e^{[A]}$ denotes the exponential of $[A]$ as defined for instance in Reference 5 (p. 51). Substituting equation (8) in equation (5b) leads to the approximated interaction force–displacement relationship

$$\{R(t)\} = \int_0^t [C]e^{[A](t-\tau)}[B]\{u(\tau)\}d\tau + [K_\infty]\{u(t)\} + [C_\infty]\{\dot{u}(t)\} \tag{9}$$

The term $[C]e^{[A]t}[B]$ represents the unit-impulse response matrix of the approximated regular part. Each term of $[S_r(t)]$ in equation (1) is thus approximated by a linear combination of complex exponential functions $e^{\lambda_i t}$ with λ_i denoting the eigenvalues of $[A]$.

A through treatment of the subject starting from the background of a structural dynamic analyst is not possible in a paper of reasonable length. It is the aim of this contribution to draw the attention of the reader to the concepts and results of the procedure. In the following, only the main steps of the balancing approximation and the corresponding realizations are described. For details, Reference 4 and in particular Sections 2.3, 2.6, 2.7, 3.1, 4.4, 4.5, 5.1, 5.3 and 5.4 should be consulted. A historical review is also contained, which is not repeated in this paper. To acquire a background in linear system theory and signal processing, the reader is referred to References 5–7.

2. DIAGONALIZATION

In principle, the methods of linear system theory can be applied directly to the total matrix $[S_r(t)]$. As the number of degrees of freedom in the nodes on the structure–medium interface is, in general, quite large, the involved huge matrices when addressing the total matrix become unmanageable when performing the rational approximation. Thus, in a typical unbounded medium–structure interaction analysis a procedure should be used which addresses each element of the matrix separately. Along this line, the so-called diagonalization procedure discussed in Reference 8 makes use, in addition of the symmetry of the matrix reducing the computational effort of the actual calculation in the time domain. As another advantage, the property matrices of the realization are symmetric.

In the diagonalization procedure the regular part of the fully coupled unit-impulse response matrix $[S_r(t)]$ of dimensions $N \times N$ is decomposed without introducing any approximation as

$$[S_r(t)] = [T]\lceil S_r^m(t) \rfloor [T]^{\mathrm{T}} \tag{10}$$

$\lceil S_r^m(t) \rfloor$, of dimensions $N(N+1)/2 \times N(N+1)/2$, denotes the diagonal matrix of the modelled unit-impulse response coefficients (superscript m for modelled). $[T]$ represents the constant transformation matrix for which various choices are possible[8]

$$\{u^m(t)\} = [T]^{\mathrm{T}}\{u(t)\} \tag{11a}$$

$$\{R_r(t)\} = [T]\{R_r^m(t)\} \tag{11b}$$

Each element of the modelled displacements $\{u^m(t)\}$ and the corresponding element of the interaction forces $\{R_r^m(t)\}$ are related by the scalar equation

$$R_r^m(t) = \int_0^t S_r^m(t-\tau)u^m(\tau)\,\mathrm{d}\tau \tag{12}$$

The rational approximation and realization as well as the transformations between continuous- and discrete-time systems are performed for each scalar diagonal element $S_r^m(t)$ independently from the others. This leads to the $N(N+1)/2$ local realizations. To construct the corresponding global realization of the regular part, the local realizations are assembled and processed as described on the right-hand side of equation (10). Merging the singular part (present in equations (1) and (4)) then yields the final global realization. In the following sections, the processing on the local level only is addressed.

3. DISCRETE-TIME SYSTEMS BY SAMPLING AND INTERPOLATION IN TIME DOMAIN

The regular part of the modelled unit-impulse response coefficients of the continuous-time system is known at distinct time stations $S_r^m(t_j)$, which corresponds to sampling with Δt. An equivalent discrete-time system is to be determined which when subjected to the sampling of the input of the continuous-time system results in a output which is a good approximation of the sampling of that of the continuous-time system. Depending on the assumed properties of the input, various discrete-time systems can be constructed which differ in their unit-pulse response. Three different kinds of inputs are addressed (Figure 2): the impulse-invariant method, the zero-order hold method and the triangle hold method.

In the *impulse-invariant method*, the unit-pulse response of the discrete-time system is set equal to $\Delta t S_r^m(t_j)$. In contrast to the other two methods addressed below, the unit-impulse response of the discrete-time system is set simply proportional to the samples of the lingering impulse response of the continuous time system.[7] Exact results are obtained when the input consists of a train of Dirac impulses or when both the unit-impulse response of the continuous-time system $S_r^m(t)$ and the input $u^m(t)$ are frequency-bandlimited with $\omega_{\max}$ below $\pi/\Delta t$ (Nyquist frequency, theorem of sampling). In the *zero-order hold method*, the unit-pulse response is obtained by convolution of the constant input between 0 and Δt (with $S_r^m(t)$ assumed to vary linearly between

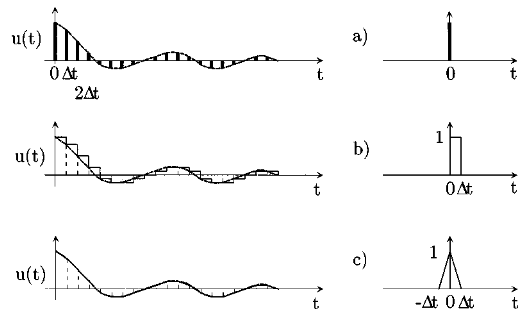

Figure 2. Assumed interpolation of sampled input: (a) impulse-invariant method; (b) zero-order hold method (piecewise constant interpolation); (c) triangle hold method (piecewise linear interpolation)

time stations) evaluated at the time stations. This method leads to exact results when the input is piecewise constant (assuming that $S_r^m(t)$ is piecewise linear). In the *triangle hold method*, the procedure is analogous whereby the convolution is performed for the triangular input between $-\Delta t$ and Δt. For piecewise linear input, this method is exact.

For harmonic input, the response for the three systems will differ. As already mentioned, in the impulse-invariant method, the exact results are obtained for $\omega < \pi/\Delta t$ with a bandlimited $S_r^m(t)$. If not, aliasing occurs. In the other two methods the smaller the frequency, the more accurate the frequency response is.

The unit-pulse response of the discrete-time system represents the so-called *Markov parameters* $h_j (j = 0, 1, \ldots 2J)$, which correspond to the partial data available ($t_{\max} = 2J\Delta t$).

4. BALANCING APPROXIMATION

The starting point to determine the matrices $[\tilde{A}]$, $\{\tilde{B}\}$, $[\tilde{C}]$ and $\tilde{D}$ of the finite-difference equation (equation (2)) are the Markov parameters h_j. The latter from $j = 1$ onwards up to $2J$ (an even number for simplicity) are arranged in the so-called *Hankel matrix* $[H]$ of dimensions $2J \times 2J$ as specified in Figure 3. The lower right part of the matrix (where the Markov parameters are not known, as partial data only are available) is set equal to zero.

The following three steps are processed. For details, the reader should consult Reference 4.

Step 1: The submatrix $[H_J]$ of dimensions $J \times J$, formed by the elements of $[H]$ located at the intersection of the first J rows and columns, is addressed (Figure 3). The *singular value decomposition* of $[H_J]$ is then performed, yielding the singular values σ_j $(j = 1, 2, \ldots J)$ which are arranged in decreasing values. The numerical rank of $[H_J]$ in a practical application equals the degree of the rational function, the first $2J$ Markov parameters of which are equal to h_j $(j = 1, 2 \ldots 2J)$.

Step 2: A tolerance *tol* is introduced, which determines the degree M of the rational approximation. M equals the smallest integer which satisfies

$$\text{tol} \sum_{j=1}^{J} \sigma_j \geqslant \sum_{j=M+1}^{J} \sigma_j \tag{13}$$

Step 3: According to the algorithm described in Section 6.5 of Reference 5, the singular value decomposition is applied to the Hankel matrix $[H]$. Only the first M largest singular value of the $2J$ values are retained.

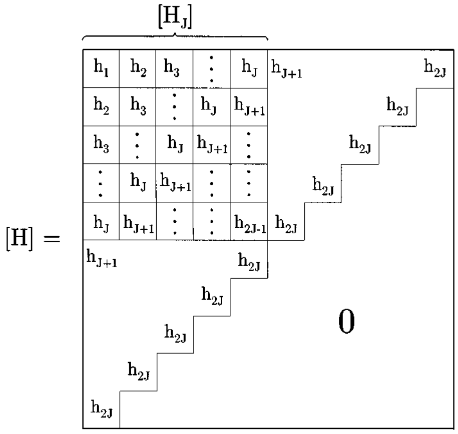

Figure 3. Hankel matrix $[H]$

The matrices $[\tilde{A}]$, $\{\tilde{B}\}$ and $[\tilde{C}]$ then follow. The scalar $\tilde{D}$ in equation (2b) is equal to h_0. This yields

$$\{x(k+1)\} = [\tilde{A}]\{x(k)\} + \{\tilde{B}\}u^m(k) \tag{14a}$$

$$R_r^m(k) = [\tilde{C}]\{x(k)\} + \tilde{D}u^m(k) \tag{14b}$$

which is the local finite-difference realization in discrete time.

This algorithm exhibits appealing features. Of the $2J$ state variables corresponding to the realization of $[H]$, the algorithm neglects those $2J - M$ state variables for which a large amount of input energy is required for their excitation and at the same time a small amount of their energy is released as output. This characterizes in loose terms the *balancing approximation*. In addition, the procedure leads to a stable realization, i.e. a bounded input will result in a bounded output. The magnitudes of the eigenvalues of $[\tilde{A}]$ are smaller than 1.

5. RECURSIVE EVALUATION OF INTERACTION FORCES

Alternatively, the M previous interaction forces and M previous displacements can be introduced as state variables. Performing the z-transformation of equation (14) and eliminating the state variables $\{x(k)\}$ yields

$$R_r^m(z) = (\tilde{D} + [\tilde{C}](z[I] - [\tilde{A}])^{-1}\{\tilde{B}\})u^m(z) \tag{15}$$

which represents the rational approximation in the z-domain. Equation (15) is straight-forwardly written as

$$R_r^m(z) = \frac{\tilde{b}_0 + \tilde{b}_1 z^{-1} + \cdots + \tilde{b}_M z^{-M}}{1 + \tilde{a}_1 z^{-1} + \cdots + \tilde{a}_M z^{-M}}\, u^m(z) \tag{16}$$

Applying the inverse z-transformation results in the local recursive evaluation of the regular part of the interaction forces in discrete time

$$R_r^m(k) = -\sum_{m=1}^{M} \tilde{a}_m R_r^m(k-m) + \sum_{m=0}^{M} \tilde{b}_m u^m(k-m) \tag{17}$$

6. CONTINUOUS-TIME SYSTEM USING IMPULSE-INVARIANT METHOD

It can be appropriate to construct starting from the realization of the discrete-time system (equation (14)) a continuous-time system. The corresponding realization can be expressed in local property matrices and thus in the same form as the structure.

Only the impulse-invariant method to construct a discrete-time system (Section 3) is examined in this section. It can be shown that the unit-impulse response function of the continuous-time system interpolates the Markov parameters of the discrete-time realization. In addition, when the selected Δt satisfies the sampling theorem, the frequency response of the discrete- and continuous-time realizations coincide up to the Nyquist frequency (Section 3).

To obtain a strictly proper rational continuous-time realization, as described by $[A]$, $\{B\}$ and $[C]$ in equation (5), the discrete-time realization with $[\tilde{A}]$, $\{\tilde{B}\}$, $[\tilde{C}]$ and $\tilde{D}$ must satisfy

$$\tilde{D} = \tfrac{1}{2}[\tilde{C}][\tilde{A}]^{-1}\{\tilde{B}\} \tag{18}$$

To meet this condition, Step 3 of Section 4 must be performed with the modified Markov parameters as described in Section 4.5.2 of Reference 4.

It can be shown that the matrices of the continuous-time realization follow from those of the discrete-time realization as

$$[A] = \frac{1}{\Delta t}\,\mathrm{Log}[\tilde{A}] \tag{19a}$$

$$\{B\} = (\Delta t[\tilde{A}])^{-1/2}\{\tilde{B}\} \tag{19b}$$

$$[C] = [\tilde{C}](\Delta t[\tilde{A}])^{-1/2} \tag{19c}$$

$\mathrm{Log}[\tilde{A}]$ denotes the principal logarithm of $[\tilde{A}]$. $\mathrm{Log}\,[\tilde{A}]$ and $[\tilde{A}]^{-1/2}$ are functions of the square matrix $[\tilde{A}]$ as defined, for instance, in Section 2.7 of Reference 5.

The local realization in continuous time is specified by the linear differential equation

$$\{\dot{x}(t)\} = [A]\{x(t)\} + \{B\}u^m(t) \tag{20a}$$

$$R_r^m(t) = [C]\{x(t)\} \tag{20b}$$

This continuous-time realization is stable. The real parts of the eigenvalues of $[A]$ are negative.

Summarising, the continuous-time system using the impulse-invariant method exhibits the following new aspects in the field of the rational approximation: (1) The procedure can easily be automated. (2) The stability of the approximation is guaranteed. (3) The approximation of the regular part is strictly proper, yielding accurate results also for high frequencies. Thus, no term $[D]\{u(t)\}$ appears in the linear differential equation of first order (equation (5b)).

7. PROPERTY MATRICES

Alternatively, the non-symmetric realization of equation (20) can be transformed to equivalent symmetric formulations of either first- or second-order systems. Two procedures are described.

The first procedure is applicable when $[A]$ can be transformed to the diagonal form which is the case in practical applications. Performing a transformation of the state variables $\{x_1\} = [\Phi]\{x\}$ with the eigenvector matrix $[\Phi]$ of $[A]$ yields in equation (20) the diagonal matrix $\lceil A_1 \rfloor = [\Phi]^{-1}[A][\Phi]$, $\{B_1\} = [\Phi]^{-1}\{B\}$ and $[C_1] = \{B_1\}^{\mathrm{T}}$. Performing the Laplace transformation and eliminating $\{x_1\}$ yields directly the partial fraction expansion which exhibits only simple poles. Applying the discrete-element models of either first- or second-order as described in detail in Reference 8 leads to the local property matrices of the first or second order. Practical examples using these physically motivated spring–dashpot–mass arrangements are discussed in the references cited in Reference 8.

In the second procedure, a symmetric first-order system is first constructed. Premultiplying equation (20a) by the diagonal matrix $\lceil \Sigma_s \rfloor$ containing the sign of the eigenvalues corresponding to the first M selected singular values of the Hankel matrix $[H]$ in Step 3 of the algorithm described in Section 4 yields

$$\begin{bmatrix} -\lceil \Sigma_s \rfloor & \{0\} \\ [0] & 0 \end{bmatrix} \begin{Bmatrix} \{\dot{x}(t)\} \\ \dot{u}^m(t) \end{Bmatrix} + \begin{bmatrix} [A_{sy}] & \{B_{sy}\} \\ \{B_{sy}\}^{\mathrm{T}} & 0 \end{bmatrix} \begin{Bmatrix} \{x(t)\} \\ u^m(t) \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ R_r^m(t) \end{Bmatrix} \tag{21}$$

with the symmetric matrix

$$[A_{sy}] = \lceil \Sigma_s \rfloor [A] \tag{22a}$$

and

$$\{B_{sy}\} = \lceil \Sigma_s \rfloor \{B\} = [C]^{\mathrm{T}} \tag{22b}$$

Equation (21) represents the local first-order symmetric realization in continuous time with the coefficient matrices of the velocities and the displacements representing the damping matrix $[C^1]$ and the static-stiffness matrix $[K^1]$, respectively (property matrices).

An equivalent second-order symmetric realization is constructed as follows. Performing the Laplace transformation of equation (21) (with the complex variable s) and eliminating the state variables $\{x(s)\}$ yields

$$R_r^m(s) = -\{B_{sy}\}^{\mathrm{T}}([A_{sy}] - s\lceil \Sigma_s \rfloor)^{-1}\{B_{sy}\}u^m(s) \tag{23}$$

Applying the matrix inversion lemma (Section 3 of the appendix in Reference 9) yields

$$([A_{sy}] - s\lceil \Sigma_s \rfloor)^{-1} = [A_{sy}]^{-1} - s[A_{sy}]^{-1}(-s\lceil \Sigma_s \rfloor + s^2[A_{sy}]^{-1})^{-1}[A_{sy}]^{-1}s \tag{24}$$

Substituting equation (24) in equation (23) leads to an equation which is identical to the Laplace transformation, after eliminating the state variables $\{w(s)\}$, of

$$\begin{bmatrix} -[A_{sy}]^{-1} & \{0\} \\ [0] & 0 \end{bmatrix} \begin{Bmatrix} \{\ddot{w}(t)\} \\ u^m(t) \end{Bmatrix} + \begin{bmatrix} \lceil \Sigma_s \rfloor & [A_{sy}]^{-1}\{B_{sy}\} \\ \{B_{sy}\}^{\mathrm{T}}[A_{sy}]^{-1} & 0 \end{bmatrix} \begin{Bmatrix} \{\dot{w}(t)\} \\ \dot{u}^m(t) \end{Bmatrix}$$
$$+ \begin{bmatrix} [0] & \{0\} \\ [0] & -\{B_{sy}\}^{\mathrm{T}}[A_{sy}]^{-1}\{B_{sy}\} \end{bmatrix} \begin{Bmatrix} \{w(t)\} \\ u^m(t) \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ R_r^m(t) \end{Bmatrix} \tag{25}$$

Note that the dimension of $\{w(t)\}$ is equal to that of $\{x(t)\}$. Equation (25) represents the local second-order symmetric realization in continuous time with the coefficient matrices of the accelerations, velocities and displacements representing the mass matrix $[M^2]$, damping matrix $[C^2]$ and static-stiffness matrix $[K^2]$, respectively.

8. IN-PLANE MOTION OF SEMI-INFINITE LAYER FIXED AT ITS BASE

The in-plane motion of a semi-infinite layer with a free and fixed boundary extending to infinity of constant depth d, shear modulus G, Poisson's ratio $\nu = \frac{1}{3}$ and mass density ρ is addressed (Figure 4). On the vertical structure–medium interface 8 line finite elements, each with 3 nodes, are introduced (not shown in the figure) in the scaled boundary finite-element method (alias consistent infinitesimal finite-element cell method) discussed in Reference 2. This discretization permits an adequate modelling up to the dimensionless frequency $a_0 = \omega d/c_s = 2{\cdot}5\pi$ ($c_s = \sqrt{G/\rho}$). To reduce the data for the examination, 4 nodes with the numbers shown in Figure 4 with piecewise linear displacements are introduced, and the corresponding reduction is performed based on virtual-work considerations. This leads to the regular part of the unit-impulse response matrix $[S_r(t_j)]$ calculated up to $t_{\max} = 2.94d/c_s$ with $\Delta t = 0.021d/c_s$ and the singular part.

The symmetric matrix $[S_r(t_j)]$ of dimensions 8×8 is diagonalized (Section 2). The resulting 36 modelled unit-impulse response coefficients are processed using the impulse-invariant method with a tolerance $tol = 5 \times 10^{-3}$ (Sections 3 and 4), leading to $[\tilde{A}]$, $\{\tilde{B}\}$, $[\tilde{C}]$ and $\tilde{D}$ in equation (14). The continuous-time system is then determined (Section 6) resulting in $[A]$, $\{B\}$ and $[C]$ in equation (20). The modelled

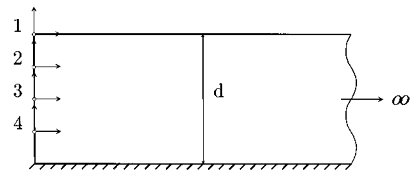

Figure 4. In-plane motion of semi-infinite layer fixed at its base with discretization on structure–medium interface

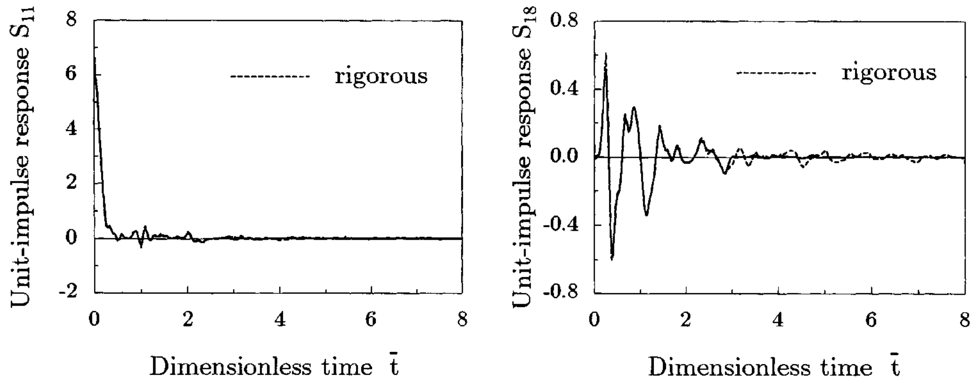

Figure 5. Regular part of unit-impulse response coefficients

unit-impulse response coefficients corresponding to this continuous-time system are calculated. Performing the transformation of equation (10) results in the regular part of the unit-impulse response matrix of the realization which is compared to the original data $[S_r(t_j)]$ denoted as rigorous. This comparison is shown, for the coefficients S_{r11} and S_{r18}, in Figure 5. The ordinates are non-dimensionalized with $d/(\rho c_s)$ and on the abscissas the non-dimensionless time $\bar{t} = tc_s/d$ is introduced. Excellent agreement results.

It is of interest to examine the behaviour in the frequency domain. The scaled boundary finite-element method can also calculate the dynamic-stiffness matrix $[S(\omega_j)]$ which is again denoted as rigorous. The dynamic-stiffness coefficients S_{11} and S_{18} corresponding to the realization are compared in Figure 6 to the corresponding coefficients of $[S(\omega_j)]$. The dynamic-stiffness coefficients are decomposed as

$$[S(a_0)] = K_{11}[k(a_0) + \mathrm{i}a_0 c(a_0)] \tag{26}$$

with the dimensionless frequency $a_0 = \omega d/c_s$ and the static-stiffness coefficient K_{11} used for non-dimensionalization. Although the frequency dependency is very strong, the agreement is excellent.

9. CONCLUDING REMARKS

1. In a practical dynamic unbounded medium–structure-interaction analysis, the global coupling in the time domain leading to convolution integrals in the rigorous interaction force–displacement relationship of the unbounded medium must be avoided to obtain computational efficiency. This can be achieved by performing a rational approximation and a realization, which corresponds to representing the unbounded medium by a finite number of degrees of freedom in the same way as the structure is.

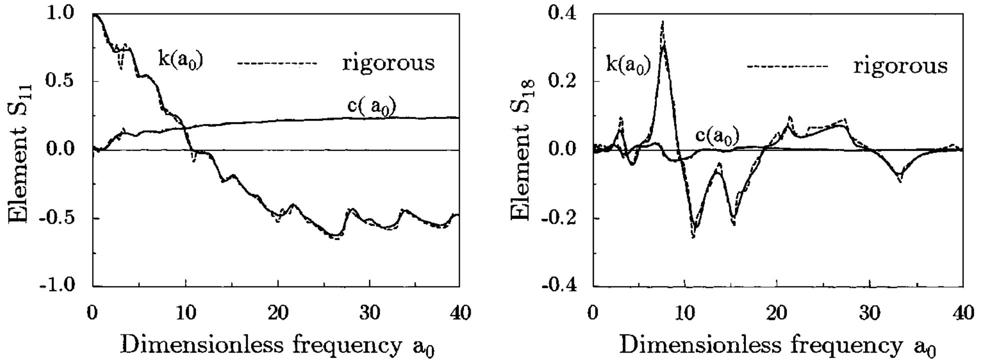

Figure 6. Dynamic-stiffness coefficients

2. Starting from the unit-impulse response matrix with the partial data, the balancing approximation leads to a discrete-time formulation such as the recursive evaluation of the interaction forces. Alternatively, a continuous-time formulation can subsequently be derived yielding, for instance, the property matrices (static-stiffness, damping and mass matrices).
3. The balancing approximation algorithm is easily automated, and stability of the realizations is guaranteed and high accuracy results.

REFERENCES

1. J. P. Wolf, *Soil–Structure-Interaction Analysis in Time Domain*, Prentice-Hall, Englewood Cliffs, NJ, 1988.
2. J. P. Wolf and C. Song, *Finite-Element Modelling of Unbounded Media*, Wiley, Chichester, 1996.
3. J. P. Wolf and C. Song, 'Unit-impulse response of unbounded medium by scaled boundary finite-element method', *Comput. Meth. Appl. Mech. Engng*., in press.
4. A. Paronesso, 'Rational approximation and realization of generalized force–displacement relationship of an unbounded medium', *Ph.D. thesis*, Laboratory of Hydraulic Constructions, Department of Civil Engineering, Swiss Federal Institute of Technology Lausanne, February 1997.
5. C. T. Chen, *Linear System Theory and Design*, Holt, Rinehart and Winston, New York, 1984.
6. G. F. Franklin, J. D. Powel and M. L. Workman, *Digital Control of Dynamic Systems*, Addison-Wesley, Reading, MA, 1990.
7. A. V. Oppenheim, A. S. Willsky and I. T. Young, *Signals and Systems*, Prentice-Hall, Englewood Cliffs, NJ, 1983.
8. A. Paronesso and J. P. Wolf, 'Property matrices identification of unbounded medium from unit-impulse response functions using Legendre polynomials: formulation', *Earthquake Engng. Struct. Dyn*. **25**, 1231–1245 (1996).
9. K. Ogata, *Discrete-Time Control Systems*, Prentice-Hall, Englewood Cliffs, NJ, 1987.

